

Ladder epochs and ladder chain of a Markov random walk with discrete driving chain

Gerold Alsmeyer

Abstract Let $(M_n, S_n)_{n \geq 0}$ be a Markov random walk with positive recurrent driving chain $(M_n)_{n \geq 0}$ having countable state space \mathcal{S} and stationary distribution π . It is shown in this note that, if the dual sequence $(\#M_n, \#S_n)_{n \geq 0}$ is positive divergent, i.e. $\#S_n \rightarrow \infty$ a.s., then the strictly ascending ladder epochs $\sigma_n^>$ of $(M_n, S_n)_{n \geq 0}$ (see (3)) are a.s. finite and the ladder chain $(M_{\sigma_n^>})_{n \geq 0}$ is positive recurrent on some $\mathcal{S}^> \subset \mathcal{S}$. We also provide simple expressions for its stationary distribution $\pi^>$, an extension of the result to the case when $(M_n)_{n \geq 0}$ is null recurrent, and a counterexample that demonstrates that $\#S_n \rightarrow \infty$ a.s. does not necessarily entail $S_n \rightarrow \infty$ a.s., but rather $\limsup_{n \rightarrow \infty} S_n = \infty$ a.s. only. Our arguments are based on Palm duality theory, coupling and the Wiener-Hopf factorization for Markov random walks with discrete driving chain.

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1 Introduction

Let $M = (M_n)_{n \geq 0}$ be a positive recurrent Markov chain, defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, with at most countable state space \mathcal{S} , transition matrix $P = (p_{ij})_{i,j \in \mathcal{S}}$ and unique stationary distribution $\pi = (\pi_i)_{i \in \mathcal{S}}$. Further, let

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$(X_n)_{n \geq 1}$ be a sequence of real-valued random variables which are conditionally independent given M and

$$\mathbb{P}(X_n \in \cdot | M) = \mathbb{P}(X_n \in \cdot | M_{n-1}, M_n) =: F_{M_{n-1}, M_n} \quad \text{a.s.}$$

for all $n \geq 1$. Equivalently, $(M_n, X_n)_{n \geq 1}$, called *Markov-modulated sequence*, forms a temporally homogeneous Markov chain on $\mathcal{S} \times \mathbb{R}$ with a special transition kernel, namely

$$\begin{aligned} Q((i, x), \{j\} \times (-\infty, t]) &:= \mathbb{P}(M_{n+1} = j, X_{n+1} \leq t | M_n = i, X_n = x) \\ &= \mathbb{P}(M_{n+1} = j, X_{n+1} \leq t | M_n = i) \\ &= p_{ij} F_{ij}(t) \end{aligned} \quad (1)$$

for all $i, j \in \mathcal{S}$, $t, x \in \mathbb{R}$ and $n \geq 1$.

As usual, we write \mathbb{P}_i for $\mathbb{P}(\cdot | M_0 = i)$, \mathbb{E}_i for expectations with respect to \mathbb{P}_i , and put $\mathbb{P}_\lambda := \sum_{i \in \mathcal{S}} \lambda_i \mathbb{P}_i$ for any measure $\lambda = (\lambda_i)_{i \in \mathcal{S}}$ on \mathcal{S} . Under \mathbb{P}_π , $(M_n, X_n)_{n \geq 1}$ forms a stationary sequence and can therefore be extended to a doubly infinite sequence $(M_n, X_n)_{n \in \mathbb{Z}}$. Note that “ \mathbb{P}_π -a.s.”, also just stated as “a.s.” hereafter, means \mathbb{P}_i -a.s. for all $i \in \mathcal{S}$ because all π_i are positive. In the doubly infinite setup, we further use $\mathbb{P}_{i,x}$ for $\mathbb{P}(\cdot | M_0 = i, X_0 = x)$ and let \mathbb{P}_ν have the obvious meaning for a measure on $\mathcal{S} \times \mathbb{R}$. Finally, the stationary distribution of $(M_n, X_n)_{n \in \mathbb{Z}}$ is denoted as ξ .

Under the stated assumptions, the additive sequence $(S_n)_{n \geq 0}$, defined by $S_0 := 0$ and $S_n := \sum_{k=1}^n X_k$ for $n \geq 1$, as well as its bivariate extension $(M_n, S_n)_{n \geq 0}$ are called a *Markov random walk (MRW)* or *Markov-additive process* and M its (*discrete*) *driving chain*. If $(S_n)_{n \geq 0}$ is *positive divergent*, i.e.

$$\lim_{n \rightarrow \infty} S_n = \infty \quad \text{a.s.}, \quad (2)$$

then the associated (*strictly ascending*) *ladder height process* may be defined as its maximal increasing subsequence $(S_{\sigma_n^>})_{n \geq 0}$ with $\sigma_0^> \equiv 0$. Here “maximal” means that any other subsequence $(S_{\eta_n})_{n \geq 0}$ with positive increments and $\eta_0 \equiv 0$ has a lower sampling rate ($\sigma_n^> \leq \eta_n$ for all $n \geq 1$). Formally, we have for $n \geq 1$

$$\sigma_n^> := \inf\{k > \sigma_{n-1}^> : S_k > S_{\sigma_{n-1}^>}\} \quad (3)$$

with the usual convention that this is ∞ if $\sigma_{n-1}^> = \infty$ or the stopping condition is never met. Using the strong Markov property, one can easily verify that

$$(M_n^>, S_n^>)_{n \geq 0} := (M_{\sigma_n^>}, S_{\sigma_n^>})_{n \geq 0} \quad \text{and} \quad (M_n^>, \sigma_n^>)_{n \geq 0}$$

form again MRW's. Their common driving chain $(M_n^>)_{n \geq 0}$ is called *ladder chain* hereafter. The main purpose of this note is to show how Palm calculus and Wiener-Hopf factorization may be employed in an elegant way to derive that the ladder chain is again positive recurrent and to provide information on its stationary distribution $\pi^>$, say, in terms of π . In the case when the

stationary drift $\mu := \mathbb{E}_\pi X_1$ exists and is positive and thus $n^{-1}S_n \rightarrow \mu$ a.s., in particular (2) holds true, the positive recurrence of $(M_n^>)_{n \geq 0}$ along with other properties of $(M_n^>, S_n^>, \sigma_n^>)_{n \geq 0}$ has already been proved in [1] even allowing for uncountable state space \mathcal{S} . On the other hand, the arguments given there are rather technical, owing to the more delicate renewal structure of a general positive Harris chain on a continuous state space as opposed to a discrete Markov chain. To keep the amount of technicalities at a minimum has been the main reason to restrict ourselves here to the discrete setting.

It should be clear that recurrence of the ladder chain and related properties form an important ingredient when dealing with Markov renewal theory or, more generally, fluctuation-theoretic properties of the MRW $(M_n, S_n)_{n \geq 0}$. Namely, it allows to identify a subsequence $(S_{\tau_n^>(s)})_{n \geq 0}$ of $(S_n^>)_{n \geq 0}$ and thus of $(S_n)_{n \geq 0}$ which is an ordinary renewal process (a RW with positive increments) because $M_{\tau_n^>(s)} = s$ for all $n \geq 1$ and some $s \in \mathcal{S}$. Such embeddings are fundamental when attempting to derive results of the aforementioned kind for MRW's by drawing on known results for ordinary RW's or renewal processes. For the case when $(M_n, S_n)_{n \geq 0}$ has positive stationary drift, this has recently been demonstrated in [3] by showing that all fundamental Markov renewal theorems can be deduced with the help of such embeddings and the use of classical renewal theory.

2 Main results

In order to present our main results, we first need to return to the doubly infinite sequence $(M_n, X_n)_{n \in \mathbb{Z}}$ under \mathbb{P}_ξ and define the associated doubly infinite random walk $(S_n)_{n \in \mathbb{Z}}$ via

$$S_n := \begin{cases} \sum_{i=1}^n X_i, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0, \\ -\sum_{i=n+1}^0 X_i, & \text{if } n \leq -1, \end{cases} \quad (4)$$

thus $S_n = S_{n-1} + X_n$ for all $n \in \mathbb{Z}$. We note that the forward sequence $(M_n, X_n)_{n \in \mathbb{Z}}$ is a stationary Markov chain with transition kernel Q given by (1), while the backward sequence $(\#M_n, \#X_n)_{n \in \mathbb{Z}} := (M_{-n}, X_{-n+1})_{n \geq 0}$ is a stationary Markov chain with the dual kernel $\#Q$ given by

$$\#Q((i, x), \{j\} \times (-\infty, t]) = \frac{\pi_j p_{ji}}{\pi_i} F_{ji}(t) \quad (5)$$

for $i, j \in \mathcal{S}$ and $x, t \in \mathbb{R}$. We call $(\#M_n, \#X_n)_{n \in \mathbb{Z}}$ and $(\#M_n)_{n \in \mathbb{Z}}$ the dual of $(M_n, X_n)_{n \in \mathbb{Z}}$ and $(M_n)_{n \in \mathbb{Z}}$, respectively. Accordingly, the MRW $(\#M_n, \#S_n)_{n \in \mathbb{Z}}$ with $\#S_n$ having the obvious meaning is called the dual of $(M_n, S_n)_{n \geq 0}$. Note that $\#S_n = -S_{-n}$ for $n \in \mathbb{Z}$.

If $\mu = \mathbb{E}_\pi X_0 > 0$, then Birkhoff's ergodic theorem implies (2) as well as the positive divergence of the dual walk, i.e.

$$\lim_{n \rightarrow \infty} \#S_n = \lim_{n \rightarrow \infty} -S_{-n} = \infty \quad \text{a.s.} \quad (6)$$

However, the last assertion may fail if only (2) holds, see Section 6 at the end of the paper for a counterexample. If (6) is assumed, then, with probability one, there is a doubly infinite sequence $(\sigma_n)_{n \in \mathbb{Z}}$ of ladder epochs determined through $\dots < \sigma_{-1} < \sigma_0 \leq 0 < \sigma_1 < \sigma_2 < \dots$ and $S_{\sigma_n} > \sup_{j < \sigma_n} S_j$ for all $n \in \mathbb{Z}$. In particular,

$$\begin{aligned} \sigma_1 &:= \inf\{k \geq 1 : S_k > \sup_{j < k} S_j\}, \\ \sigma_0 &:= \sup\{k \leq 0 : S_k > \sup_{j < k} S_j\}. \end{aligned}$$

The reader should notice that $(\sigma_n)_{n \geq 1}$ and $(\sigma_n^>)_{n \geq 1}$ are generally different although these sequences share the same recursive structure (see (3)):

$$\sigma_n = \inf\{k > \sigma_{n-1} : S_k - S_{\sigma_{n-1}} > 0\}.$$

Our main result can be viewed as a specialization of a similar result stated (without proof) by Lalley [7, Section 4B] for general random walks with integrable stationary increments.

Theorem 2.1 *Let $(M_n, S_n)_{n \geq 0}$ be a MRW having positive recurrent discrete driving chain $(M_n)_{n \geq 0}$ with state space \mathcal{S} , transition matrix $P = (p_{ij})_{i,j \in \mathcal{S}}$ and stationary distribution $\pi = (\pi_i)_{i \in \mathcal{S}}$. Suppose that the dual $(\#S_n)_{n \geq 0}$ is positive divergent in the sense of (6). Then the ladder chain $M^> = (M_n^>)_{n \geq 0}$ possesses the unique stationary distribution $\pi^> = (\pi_i^>)_{i \in \mathcal{S}}$, defined by*

$$\pi_i^> := \frac{1}{c} \mathbb{E}_\xi \left(\frac{1}{\sigma_1 - \sigma_0} \mathbf{1}_{\{M_{\sigma_0} = i\}} \right) = \frac{1}{c} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0), \quad (7)$$

and has π -density $f(i) = c^{-1} \mathbb{P}_\xi(\sigma_0 = 0 | M_0 = i)$ for $i \in \mathcal{S}$. Here ξ equals the stationary law of $(M_n, X_n)_{n \in \mathbb{Z}}$ and

$$c := \mathbb{E}_\xi \left(\frac{1}{\sigma_1 - \sigma_0} \right) = \mathbb{P}_\xi(\sigma_0 = 0) \in (0, 1].$$

Moreover, the ladder chain is positive recurrent on $\mathcal{S}^> := \{i \in \mathcal{S} : \pi_i^> > 0\}$, and

$$\mathbb{P}_i(\tau^>(\mathcal{S}^>) < \infty) = 1 \quad (8)$$

for all $i \in \mathcal{S}$, where $\tau^>(\mathcal{S}^>) := \inf\{n \geq 1 : M_n^> \in \mathcal{S}^>\}$.

Remark 2.2 As a consequence of Theorem 2.1, we see that (6) entails the existence of an ordinary two-sided subsequence $(S_{\tau_n})_{n \in \mathbb{Z}}$ having positive iid

increments (choose the τ_n as those ladder epochs with $\tau_0 \leq 0 < \tau_1$ having further $M_{\tau_n} = s$ for some fixed $s \in \mathcal{S}^>$) and therefore $\limsup_{n \rightarrow \infty} S_n = \infty$ a.s. which is weaker than positive divergence of $(S_n)_{n \geq 0}$.

Remark 2.3 Let us further point out that (6) is sufficient but not necessary for the positive recurrence of the ladder chain on some $\mathcal{S}^> \subset \mathcal{S}$. As an example consider a MRW $(M_n, S_n)_{n \geq 0}$ with positive recurrent driving chain such that, for some unique $s \in \mathcal{S}$, the F_{is} , $i \in \mathcal{S}$, are concentrated on $\mathbb{R}_>$, while all other F_{ij} are concentrated on $\mathbb{R}_<$. It is not difficult to further arrange for stationary drift $\mu = \mathbb{E}_\pi X_1 = 0$ and nontriviality in the sense that $\mathbb{P}_\pi(X_1 = g(M_1) - g(M_0)) < 1$ for any function $g : \mathcal{S} \rightarrow \mathbb{R}$, giving

$$\liminf_{n \rightarrow \infty} S_n = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} S_n = \infty \quad \text{a.s.}$$

(see [2, Thms. 2 and 3] or [4]) and thus the same for its dual $(\#S_n)_{n \geq 0}$. On the other hand, since $X_{\sigma_n^>}$ must be positive for each n , we find that $M_n^> = s$ for all $n \geq 1$ and so the ladder chain is trivially positive recurrent on $\mathcal{S}^> = \{s\}$.

The following simple corollary provides an alternative definition of the $\pi_i^>$ with the help of the weakly descending ladder epochs of the dual MRW $(\#M_n, \#S_n)_{n \geq 0}$, defined by $\# \sigma_0^\leq := 0$ and

$$\# \sigma_n^\leq := \inf \{k > \# \sigma_{n-1}^\leq : \# S_k \leq \# S_{\# \sigma_{n-1}^\leq}\}$$

for $n \geq 1$, where $\inf \emptyset := \infty$ and $\# \sigma^\leq := \# \sigma_1^\leq$.

Corollary 2.4 *Under the assumptions of Theorem 2.1, it further holds that*

$$\pi_i^> = \frac{1}{c} \pi_i \mathbb{P}_i(\# \sigma^\leq = \infty) = \frac{1}{c} \mathbb{P}_\pi(\# M_0^\leq = i, \# \sigma^\leq = \infty) \quad (9)$$

for all $i \in \mathcal{S}$, giving

$$c := \mathbb{P}_\pi(\# \sigma^\leq = \infty)$$

and

$$\mathcal{S}^> = \{i \in \mathcal{S} : \mathbb{P}_i(\# \sigma^\leq = \infty) > 0\}. \quad (10)$$

Proof. It suffices to point out that

$$\begin{aligned} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0) &= \mathbb{P}_\xi\left(M_0 = i, \max_{n \geq 1} S_{-n} < 0 = S_0\right) \\ &= \mathbb{P}_\xi\left(\# M_0 = i, \min_{n \geq 1} \# S_n > 0\right) \\ &= \pi_i \mathbb{P}_i(\# \sigma^\leq = \infty) \end{aligned}$$

for all $i \in \mathcal{S}$. \square

Instead of the second equality in (7), we will actually verify the more general one

$$\frac{1}{cm} \mathbb{P}_\xi(M_{\sigma_0} = i, \sigma_1 - \sigma_0 = m) = \frac{1}{c} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0, \sigma_1 = m) \quad (11)$$

for all $(i, m) \in \mathcal{S} \times \mathbb{N}$ in the proof of Theorem 2.1 below. Using this identity and the obvious fact that $(M_n^>, \sigma_{n+1}^> - \sigma_n^>)_{n \geq 0}$ is a stationary Markov chain under $\mathbb{P}_{\pi^>}$, we are led to the following corollary.

Corollary 2.5 *Under the assumptions of Theorem 2.1, it further holds that the law $\nu = (\nu_{i,m})_{(i,m) \in \mathcal{S} \times \mathbb{N}}$ of $(M_n^>, \sigma_{n+1}^> - \sigma_n^>)$ under $\mathbb{P}_{\pi^>}$ is given by*

$$\nu_{i,m} = \frac{1}{cm} \mathbb{P}_\xi(M_{\sigma_0} = i, \sigma_1 - \sigma_0 = m) \quad (12)$$

In particular,

$$\mathbb{P}_{\pi^>}(\sigma^> = m) = \sum_{i \in \mathcal{S}} \nu_{i,m} = \frac{\mathbb{P}_\xi(\sigma_1 - \sigma_0 = m)}{cm} \quad (13)$$

and

$$\mathbb{E}_{\pi^>} \sigma^> = \frac{1}{\mathbb{P}_\pi(\#\sigma^{\leq} = \infty)}. \quad (14)$$

Proof. We have that

$$\begin{aligned} \nu_{i,m} &= \mathbb{P}_{\pi^>}(M_0^> = i, \sigma^> = m) \\ &= \pi_i^> \mathbb{P}_i(\sigma^> = m) \\ &= \frac{1}{c} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0) \mathbb{P}_i(\sigma^> = m) \\ &= \frac{1}{c} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0, \sigma_1 = m) \\ &= \frac{1}{cm} \mathbb{P}_\xi(M_{\sigma_0} = i, \sigma_1 - \sigma_0 = m) \end{aligned}$$

as claimed. The remaining assertions are now obvious. \square

Since $\pi_i^> \mathbb{E}_i \sigma^> \leq \mathbb{E}_{\pi^>} \sigma^>$ and $\pi_i^> = c^{-1} \pi_i \mathbb{P}_i(\#\sigma^{\leq} = \infty)$ for any $i \in \mathcal{S}$, identity (14) further provides us with

$$\mathbb{E}_i \sigma^> \leq \frac{1}{\pi_i \mathbb{P}_i(\#\sigma^{\leq} = \infty)} \quad (15)$$

for all $i \in \mathcal{S}$ with the right-hand side being finite only for $i \in \mathcal{S}^>$.

A direct proof of Theorem 2.1 will be presented in the following section, but its most critical part, namely the existence of $\pi^>$, could also be deduced by drawing on a more general duality result from Palm calculus as described

in the monographies by Thorisson [9, Ch. 8] and Sigman [8]. We refer to Remark 3.1 for a sketch of details. Yet another proof via the Wiener-Hopf factorization for MRW's is briefly shown in Section 4, followed by a short treatment of the case when the driving chain is null recurrent in Section 5, see Theorem 5.1 there.

3 Proof of Theorem 2.1

We start by showing equality of the two expressions for $\pi_i^>$ in (7). We have

$$\begin{aligned}
& \mathbb{E}_\xi \left(\frac{1}{\sigma_1 - \sigma_0} \mathbf{1}_{\{M_{\sigma_0}=i\}} \right) \\
&= \sum_{k \geq 0} \sum_{l > k} \frac{1}{l} \mathbb{P}_\xi(M_{-k} = i, \sigma_0 = -k, \sigma_1 - \sigma_0 = l) \\
&= \sum_{k \geq 0} \sum_{l > k} \frac{1}{l} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0, \sigma_1 - \sigma_0 = l) \\
&= \sum_{l \geq 1} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0, \sigma_1 - \sigma_0 = l) \sum_{k=0}^{l-1} \frac{1}{l} \\
&= \sum_{l \geq 1} \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0, \sigma_1 - \sigma_0 = l) \\
&= \mathbb{P}_\xi(M_0 = i, \sigma_0 = 0),
\end{aligned}$$

where the third line follows from the stationarity of $(M_n, X_n)_{n \in \mathbb{Z}}$ (under \mathbb{P}_ξ) which provides us with

$$\begin{aligned}
& \{M_{-k} = i, \sigma_0 = -k, \sigma_1 - \sigma_0 = l\} \\
&= \left\{ M_{-k} = i, S_{-k} > \max_{j > k} S_{-j}, \max_{1 \leq j < l} S_{-k+j} \leq S_{-k}, S_{-k+l} > \max_{0 \leq j < l} S_{-k+j} \right\} \\
&= \left\{ M_{-k} = i, \min_{j > k} \sum_{n=-j}^{-k} X_n > 0, \max_{1 \leq j < l} \sum_{n=-k+1}^{-k+j} X_n \leq 0, \min_{1 \leq j \leq l} \sum_{n=-k+j}^{-k+l} X_n > 0 \right\} \\
&\stackrel{d}{=} \left\{ M_0 = i, \min_{j > 0} \sum_{n=-j}^0 X_n > 0, \max_{1 \leq j < l} \sum_{n=1}^j X_n \leq 0, \min_{1 \leq j \leq l} \sum_{n=j}^l X_n > 0 \right\} \\
&= \{M_0 = i, \sigma_0 = 0, \sigma_1 - \sigma_0 = l\}
\end{aligned}$$

for all $k \geq 0$. Here $A \stackrel{d}{=} B$ means that $\mathbf{1}_A$ and $\mathbf{1}_B$ have the same law.

Since

$$\pi_i^> = \frac{1}{c} \mathbb{P}_\xi(\sigma_0 = 0 | M_0 = i) \mathbb{P}_\xi(M_0 = i) = \frac{1}{c} \mathbb{P}_i(\sigma_0 = 0) \pi_i$$

we also obtain the asserted form of the π -density von $\pi^>$.

The next step is to verify that $\pi^>$ defines a stationary distribution for $(M_n^>)_{n \in \mathbb{Z}}$. By a similar argument as before, we obtain

$$\begin{aligned} \mathbb{P}_{\pi^>}(M_1^> = j) &= \sum_{i \in \mathcal{S}} \pi_i^> \mathbb{P}_i(M_1^> = j) \\ &= \frac{1}{c} \sum_{i \in \mathcal{S}} \mathbb{P}_\xi(\sigma_0 = 0, M_0 = i) \mathbb{P}_i(M_1^> = j) \\ &= \frac{1}{c} \sum_{i \in \mathcal{S}} \sum_{n \geq 1} \mathbb{P}_\xi(\sigma_0 = 0, \sigma_1 = n, M_0 = i, M_n = j) \\ &= \frac{1}{c} \sum_{i \in \mathcal{S}} \sum_{n \geq 1} \mathbb{P}_\xi(\sigma_{-1} = -n, \sigma_0 = 0, M_{-n} = i, M_0 = j) \\ &= \frac{1}{c} \sum_{i \in \mathcal{S}} \mathbb{P}_\xi(M_{-1}^> = i, \sigma_0 = 0, M_0 = j) \\ &= \frac{1}{c} \mathbb{P}_\xi(\sigma_0 = 0, M_0 = j) = \pi_j^> \end{aligned}$$

for all $j \in \mathcal{S}$, and this yields the desired result, for $(M_n^>)_{n \in \mathbb{Z}}$ is a Markov chain under \mathbb{P}_ξ .

Clearly, all states in $\mathcal{S}^>$ are positive recurrent for the ladder chain. A coupling argument will now be used to establish the remaining assertions including uniqueness of $\pi^>$ as a stationary distribution of $(M_n^>)_{n \geq 0}$. On a possibly enlarged probability space with underlying probability measure \mathbb{P} , let $(M'_n, X'_n)_{n \geq 0}$ and $(M''_n, X''_n)_{n \geq 0}$ be two Markov-modulated sequences with the same transition kernel as $(M_n, X_n)_{n \geq 0}$ and initial conditions

$$(M'_0, M''_0) = (i, j) \quad \text{and} \quad X'_0 = X''_0 = 0$$

for arbitrarily fixed $i, j \in \mathcal{S}$. As usual, the associated RW's are denoted by $(S'_n)_{n \geq 0}$ and $(S''_n)_{n \geq 0}$, the corresponding strictly ascending ladder epochs by σ'_n and σ''_n , respectively, where $\sigma'_0 = \sigma''_0 := 0$.

Let T be the \mathbb{P} -a.s. finite coupling time of $(M'_n, M''_n)_{n \geq 0}$, thus

$$T = \inf\{n \geq 0 : M'_n = M''_n\},$$

and put $Y' := \max_{0 \leq n \leq T} S'_n$, $Y'' := \max_{0 \leq n \leq T} S''_n$ and $Y := Y' \vee Y''$. Then the coupling process

$$(\widehat{M}_n, \widehat{X}_n) := \begin{cases} (M'_n, X'_n), & \text{falls } n \leq T, \\ (M''_n, X''_n), & \text{falls } n > T, \end{cases}$$

forms a copy of $(M'_n, X'_n)_{n \geq 0}$ and coincides with $(M''_n, X''_n)_{n \geq 0}$ after T . Moreover, $\widehat{S}_n = S'_n$ for $n \leq T$ and $\widehat{S}_n = S''_n + (S'_T - S''_T)$ for $n > T$. Denoting by $(\widehat{\sigma}_n)_{n \geq 0}$ the sequence of strictly ascending ladder epochs of $(\widehat{S}_n)_{n \geq 0}$, the crucial observation now is that the random sets $\{\sigma''_n, n \geq 0\}$ and $\{\widehat{\sigma}_n, n \geq 0\}$ a.s. coincide up to finitely many elements. Namely, if

$$\begin{aligned}\tau &:= \inf\{n : S''_{\sigma''_n} > Y + (S'_T - S''_T)^+\}, \\ \rho &:= \inf\{n : \widehat{S}_{\widehat{\sigma}_n} > Y + (S'_T - S''_T)^+\},\end{aligned}$$

then $\sigma''_\tau = \widehat{\sigma}_\rho > T$ and thus $\sigma''_{\tau+n} = \widehat{\sigma}_{\rho+n}$ for all $n \geq 1$ because $(S''_n)_{n \geq 0}$ and $(\widehat{S}_n)_{n \geq 0}$ have the same increments after T .

To complete the proof, put $M_n''^> := M''_{\sigma''_n}$, $\widehat{M}_n^> := \widehat{M}_{\widehat{\sigma}_n}$ for $n \geq 0$ and notice that

$$(M_n''^>)_{n \geq \tau} = (\widehat{M}_n^>)_{n \geq \rho}$$

for any choice of $i, j \in \mathcal{S}$. Consequently,

$$\begin{aligned}\mathbb{P}_i(M_n^> = j \text{ i.o.}) &= \mathbb{P}(\widehat{M}_n^> = j \text{ i.o.}) = \mathbb{P}(\widehat{M}_{\rho+n}^> = j \text{ i.o.}) \\ &= \mathbb{P}(M_{\tau+n}''^> = j \text{ i.o.}) = \mathbb{P}(M_n''^> = j \text{ i.o.}) \\ &= \mathbb{P}_j(M_n^> = j \text{ i.o.})\end{aligned}$$

for all $i \in \mathcal{S}$ and $j \in \mathcal{S}^>$ which shows the irreducibility of the ladder chain on $\mathcal{S}^>$ as well as (20). \square

Remark 3.1 Let us briefly describe how the Theorem 2.1 fits into the framework of Palm calculus as laid out in [9, Ch. 8]. Our starting point is here the stationary sequence $(M_n, X_{\leq n})_{n \in \mathbb{Z}}$ under \mathbb{P}_ξ , where $X_{\leq n} := (X_k)_{k \leq n}$. Notice that

$$D_m := S_m - \max_{k \leq m} S_k = \min_{k \leq m} \sum_{j=k+1}^m X_j$$

is a functional of $X_{\leq m}$ which equals 0 iff m is a strictly ascending ladder epoch of the associated doubly infinite MRW $(M_n, S_n)_{n \in \mathbb{Z}}$. In other words, the sequence $\mathbf{S} := (\sigma_n)_{n \in \mathbb{Z}}$ of ladder epochs may be viewed as the sequence of return times to $\mathcal{S} \times \{0\}$ of the stationary sequence $\mathbf{Z} := (M_n, D_n)_{n \in \mathbb{Z}}$, and it is a functional of it. For $n \in \mathbb{Z}$, define the two-sided shift

$$\theta^n((z_k)_{k \in \mathbb{Z}}, (t_k)_{k \in \mathbb{Z}}) := ((z_{n+k})_{k \in \mathbb{Z}}, (t_{n,k})_{k \in \mathbb{Z}})$$

for $n \in \mathbb{Z}$, $(z_k)_{k \in \mathbb{Z}} \in (\mathcal{S} \times \mathbb{R}_{\leq})^{\mathbb{Z}}$ and strictly increasing sequences $(t_k)_{k \in \mathbb{Z}}$ such that

$$-\infty \leftarrow \dots < t_{-1} \leq 0 < t_1 < \dots \rightarrow \infty$$

and $(t_{n,k})_{k \in \mathbb{Z}}$ equals the sequence $(t_k + n)_{k \in \mathbb{Z}}$ modulo relabeling so as to have $t_{n,0} \leq 0 < t_{n,1}$ (see also [9, p. 251]). Then the above considerations imply that the sequence

$$(\theta^n(\mathbf{Z}, \mathbf{S}))_{n \in \mathbb{Z}}$$

is \mathbb{P}_π -stationary, and the Palm duality theory [9, Theorem 8.4.1] now tells us that its cycles

$$C_n := (\sigma_{n+1} - \sigma_n, (M_k, D_k)_{\sigma_n \leq k < \sigma_{n+1}}), \quad n \in \mathbb{Z},$$

and the sequence $(M_{\sigma_n}, \sigma_{n+1} - \sigma_n)_{n \in \mathbb{Z}}$ in particular are stationary under the probability measure \mathbb{P}_ξ^0 , defined by

$$\mathbb{P}_\xi^0(dx) := \frac{1}{c(\sigma_1 - \sigma_0)} \mathbb{P}_\xi(dx)$$

with c as in Theorem 2.1 and satisfying $\mathbb{P}_\pi^0(\sigma_0 = 0) = 1$. Consequently,

$$\pi_i^> = \mathbb{P}_\xi^0(M_0 = i) = \frac{1}{c} \mathbb{E}_\xi \left(\frac{1}{\sigma_1 - \sigma_0} \mathbf{1}_{\{M_{\sigma_0} = i\}} \right), \quad i \in \mathcal{S}$$

is a stationary distribution for the ladder chain as asserted in our theorem.

4 An alternative approach via Wiener-Hopf factorization

That $\pi^>$ as defined in (9) forms a stationary distribution of the ladder chain, may also be derived with the help of the Wiener-Hopf factorization for MRW's as we will briefly demonstrate after recalling some necessary facts about this factorization with reference to Asmussen [5] and [6, p. 314ff]. Putting $(\#M_n^{\leq}, \#S_n^{\leq}) := (\#M_{\#\sigma_n^{\leq}}, \#S_{\#\sigma_n^{\leq}})$, we define the matrices

$$G := (G_{ij})_{i,j \in \mathcal{S}}, \quad G^> := (G_{ij}^>)_{i,j \in \mathcal{S}}, \quad \#G := (\#G_{ij})_{i,j \in \mathcal{S}}, \quad \#G := (\#G_{ij})_{i,j \in \mathcal{S}}$$

with measure-valued entries by

$$\begin{aligned} G_{ij} &:= p_{ij} F_{ij} = \mathbb{P}_i(M_1 = j, X_1 \in \cdot), \\ G_{ij}^> &:= \mathbb{P}_i(M_1^> = j, S_1^> \in \cdot, \sigma^> < \infty), \\ \#G_{ij}^{\leq} &:= \mathbb{P}_i(\#M_1^{\leq} = j, \#S_1^{\leq} \in \cdot, \#\sigma^{\leq} < \infty), \\ *G_{ij}^{\leq} &:= \frac{\pi_j}{\pi_i} \#G_{ji}^{\leq}. \end{aligned}$$

The convolution of matrices $A = (A_{ij})_{i,j \in \mathcal{S}}, B = (B_{ij})_{i,j \in \mathcal{S}}$ with measure-valued entries is defined in the usual manner by replacing ordinary multiplication with convolution of measures, thus $A * B = (\sum_{k \in \mathcal{S}} A_{ik} * B_{kj})_{i,j \in \mathcal{S}}$.

The following result, stated here for reference, provides the *Wiener-Hopf factorization of a MRW with discrete driving chain* and is Theorem 4.1 in [5].

Proposition 4.1 *Let $(M_n, S_n)_{n \geq 0}$ be a MRW with positive recurrent discrete driving chain $(M_n)_{n \geq 0}$. Then*

$$\delta_0 I - G = (\delta_0 I - {}^*G^{\leq}) * (\delta_0 I - G^>) \quad (16)$$

or, equivalently,

$$G = {}^*G^{\leq} + G^> - {}^*G^{\leq} * G^>, \quad (17)$$

where I denotes the identity matrix on \mathcal{S} .

Let us also state the following lemma in which $\|\nu\|$ denotes the total mass of a measure ν .

Lemma 4.2 *Under the assumptions of Theorem 2.1, the matrix $\|{}^{\#}G^{\leq}\| := (\|{}^{\#}G_{ij}^{\leq}\|)_{i,j \in \mathcal{S}}$ is truly substochastic, i.e.*

$$\sum_{j \in \mathcal{S}} \|{}^{\#}G_{ij}^{\leq}\| \leq 1$$

for all $i \in \mathcal{S}$ with strict inequality for at least one i . Furthermore,

$$\sum_{i \in \mathcal{S}} \pi_i \|{}^*G_{ij}^{\leq}\| = \pi_j \mathbb{P}_j(\# \sigma^{\leq} < \infty) \leq \pi_j \quad (18)$$

for all $j \in \mathcal{S}$ with strict inequality for at least one j .

Proof. By the definition of ${}^{\#}G_{ij}^{\leq}$, we have

$$\sum_{j \in \mathcal{S}} \|{}^{\#}G_{ij}^{\leq}\| = \sum_{j \in \mathcal{S}} \mathbb{P}_i(\# M_1^{\leq} = j, \# \sigma^{\leq} < \infty) = \mathbb{P}_i(\# \sigma^{\leq} < \infty) \leq 1$$

for all $i \in \mathcal{S}$. Moreover, strict inequality must hold for at least one i , for otherwise $\mathbb{P}_\pi(\# \sigma^{\leq} < \infty) = 1$ would follow and then inductively $\mathbb{P}_\pi(\# \sigma_n^{\leq} < \infty) = 1$ for all $n \in \mathbb{N}$, i.e.

$$\mathbb{P}_\pi(\# S_n \leq 0 \text{ i.o.}) = 1,$$

which is impossible under the assumption of two-sided positive divergence.

For the proof of (18), we note that

$$\begin{aligned} \sum_{i \in \mathcal{S}} \pi_i \|{}^*G_{ij}^{\leq}\| &= \sum_{i \in \mathcal{S}} \pi_j \|{}^{\#}G_{ji}^{\leq}\| \\ &= \pi_j \sum_{i \in \mathcal{S}} \mathbb{P}_j(\# M_1^{\leq} = i, \# \sigma^{\leq} < \infty) \\ &= \pi_j \mathbb{P}_j(\# \sigma^{\leq} < \infty), \end{aligned}$$

and by an analogous argument as before this value must be less than π_j for at least one j if two-sided positive divergence holds. \square

Proof (of the stationarity of $\pi^>$ given by (9)). It follows from the Wiener-Hopf factorization (16) that

$$I - \|G\| = (I - \|{}^*G^{\leq}\|) (I - \|G^>\|). \quad (19)$$

Multiplying this identity from the left with π^\top (the transpose of π) and observing that $\|G\| = P = (p_{ij})_{i,j \in \mathcal{S}}$ is the transition matrix of M , we infer

$$0 = \pi^\top (I - P) = \pi^\top (I - \|{}^*G^{\leq}\|) (I - \|G^>\|).$$

By Lemma 4.2, in particular (18), and the fact that all π_i are positive, the nonnegative vector

$$\pi^\top (I - \|{}^*G^{\leq}\|) = (\pi_i \mathbb{P}_i(\#\sigma^{\geq} = \infty))_{i \in \mathcal{S}}$$

is not identically zero and thus a proper solution to the equation $x(I - \|G^>\|) = 0$. After normalization through c , it therefore forms a stationary distribution of $M^>$ with transition matrix $\|G^>\|$. \square

5 The null recurrent case

It is not difficult to extend Theorem 2.1 to the case when, ceteris paribus, the driving chain $(M_n)_{n \geq 0}$ is null recurrent with essentially unique (up to positive scalars) stationary measure π . First of all, it should be observed that the ladder chain $(M_n^>)_{n \geq 0}$ may still be positive recurrent on some $\mathcal{S}^>$. In fact, $(M_n^>)_{n \geq 0}$ takes only values in the set

$$\mathcal{S}^+ := \{s \in \mathcal{S} : F_{is}(\mathbb{R}_>) > 0 \text{ for some } i \in \mathcal{S}\}$$

positive recurrence on some $\mathcal{S}^> \subset \mathcal{S}^+$ follows whenever \mathcal{S}^+ is finite (see also Remark 2.3).

The following theorem is proved in essentially the same manner as Theorem 2.1, and we therefore restrict ourselves to some comments regarding its proof. Put

$$\xi := \mathbb{P}_\pi((M_1, X_1) \in \cdot),$$

which is the essentially unique stationary measure of $(M_n, X_n)_{n \geq 0}$. Since π and ξ have infinite mass now, we define

$$c = \begin{cases} \mathbb{E}_\xi(\sigma_1 - \sigma_0)^{-1} = \mathbb{P}_\xi(\sigma_0 = 0), & \text{if these expressions are finite,} \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 5.1 *Let $(M_n, S_n)_{n \geq 0}$ be a MRW having null recurrent discrete driving chain $(M_n)_{n \geq 0}$ with essentially unique stationary measure $\pi = (\pi_i)_{i \in \mathcal{S}}$. Suppose that the dual $(\#S_n)_{n \geq 0}$ is positive divergent in the sense of (6). Then the ladder chain $M^> = (M_n^>)_{n \geq 0}$ possesses an essentially unique stationary measure $\pi^> = (\pi_i^>)_{i \in \mathcal{S}}$, defined by (7) with c as above and has π -density $f(i) = \mathbb{P}_\xi(\sigma_0 = 0 | M_0 = i)$ for $i \in \mathcal{S}$. Moreover, the ladder chain is recurrent on $\mathcal{S}^> := \{i \in \mathcal{S} : \pi_i^> > 0\}$, and*

$$\mathbb{P}_i(\tau^>(\mathcal{S}^>) < \infty) = 1 \quad (20)$$

for all $i \in \mathcal{S}$, where $\tau^>(\mathcal{S}^>) := \inf\{n \geq 1 : M_n^> \in \mathcal{S}^>\}$. Finally, positive recurrence holds iff

$$\mathbb{E}_\xi\left(\frac{1}{\sigma_1 - \sigma_0}\right) = \mathbb{P}_\xi(\sigma_0 = 0) < \infty. \quad (21)$$

Proof. If (21) is valid, then $\pi^>$ forms again a probability distribution. If the condition fails, then observe that

$$\pi_i^> = \mathbb{P}_\xi(M_0 = i, \sigma_0 = i) \leq \mathbb{P}_\xi(M_0 = i) \leq \pi_i$$

for all $i \in \mathcal{S}$ that $\pi^>$ is still a σ -finite measure. After these observations the proof follows exactly the same lines as for Theorem 2.1 and can therefore be omitted. \square

Since the dual chain $(\#M_n, \#X_n)_{n \geq 0}$ with kernel $\#Q$ given by (5) is still well-defined, we see that the assertions of Corollary 2.4 remain true as well when defining $c := 1$ in the case when (21) fails and thus $\mathbb{P}_\pi(\#\sigma^\leq = \infty) = \infty$.

6 A counterexample

The following counterexample, taken from [4] and discussed in greater detail there, shows that $(S_n)_{n \geq 0}$ and $(\#S_n)_{n \geq 0}$ may be of different fluctuation type.

Let $(M_n)_{n \geq 0}$ be a Markov chain on the set \mathbb{N}_0 of nonnegative integers which, when in state 0, picks an arbitrary $i \in \mathbb{N}$ with positive probability p_{0i} and jumps back to 0, otherwise, thus $p_{i0} = 1$. If we figure the $i \in \mathbb{N}$ being placed on a circle around 0, the transition diagram of this chain looks like a *flower with infinitely many petals*, each of the petals representing a transition from 0 to some i and back. With all p_{0i} being positive, the chain is clearly irreducible and positive recurrent with stationary probabilities $\pi_0 = \frac{1}{2}$ and

$$\pi_i = \frac{1}{2} \mathbb{E}_0 \left(\sum_{n=0}^{\tau(0)-1} \mathbf{1}[M_n = i] \right) = \frac{1}{2} \mathbb{P}_0(M_1 = i) = \frac{p_{0i}}{2}.$$

In fact, under \mathbb{P}_0 , the chain consists of independent random variables which are 0 for even n and i.i.d. for odd n with common distribution $(p_{0i})_{i \geq 1}$.

Next, we define the X_n by

$$X_n := \begin{cases} -p_{0i}^{-1}, & \text{if } M_{n-1} = 0, M_n = i, \\ 2 + p_{0i}^{-1}, & \text{if } M_{n-1} = i, M_n = 0 \end{cases}$$

for $n \geq 1$, i.e. $F_{0i} = \delta_{-p_{0i}^{-1}}$ and $F_{i0} = \delta_{2+p_{0i}^{-1}}$. It follows that

$$S_n := \begin{cases} n - 1 - p_{0M_n}^{-1}, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even} \end{cases} \quad \mathbb{P}_0\text{-a.s.},$$

and thereupon that $(M_n, S_n)_{n \geq 0}$ is oscillating, for

$$\lim_{n \rightarrow \infty} \frac{S_{2n}}{2n} = 1$$

and

$$\liminf_{n \rightarrow \infty} S_{2n+1} = \liminf_{n \rightarrow \infty} \left(n - 1 - \frac{1}{p_{0M_{2n+1}}} \right) = -\infty \quad \mathbb{P}_0\text{-a.s.}$$

The last assertion follows from the fact that, for any $a > 0$,

$$\sum_{n \geq 0} \mathbb{P}_0 \left(\frac{1}{p_{0M_{2n+1}}} > an \right) = \sum_{n \geq 0} \mathbb{P}_0(X_1 > an) \geq \frac{\mathbb{E}_0 X_1}{a} = \infty$$

and an appeal to the Borel-Cantelli lemma, giving

$$\mathbb{P}_0 \left(\frac{1}{p_{0M_{2n+1}}} > an \text{ i.o.} \right) = 1. \quad (22)$$

Turning to the dual MRW $(\#M_n, \#S_n)_{n \geq 0}$, it has increments

$$\#X_n := \begin{cases} 2 + p_{0i}^{-1}, & \text{if } \#M_{n-1} = 0, \#M_n = i, \\ -p_{0i}^{-1}, & \text{if } \#M_{n-1} = i, \#M_n = 0 \end{cases}$$

for $n \geq 1$ and is therefore positive divergent, for

$$\#S_n := \begin{cases} n + 1 + p_{0M_n}^{-1}, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even} \end{cases} \quad \mathbb{P}_0\text{-a.s.}$$

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